Year 13 Calculus EAS Workbook

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Contents

Complex Numbers 3.5

This achievement standard involves applying the algebra of complex numbers in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply the algebra of complex numbers in solving problems.	• Apply the algebra of complex numbers, using relational thinking, in solving problems.	• Apply the algebra of complex numbers, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives:
  - manipulate complex numbers and present them graphically
  - form and use polynomial, and other non-linear equations

  in the Mathematics strand of the Mathematics and Statistics Learning Area.

- Apply the algebra of complex numbers in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.

- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and relating findings to a context, or communicating thinking using appropriate mathematical statements.

- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

  and also using correct mathematical statements, or communicating mathematical insight.

- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

- Methods include a selection from those related to:
  - quadratic and cubic equations with complex roots
  - Argand diagrams
  - polar and rectangular forms
  - manipulation of surds
  - manipulation of complex numbers
  - loci
  - De Moivre's theorem
  - equations of the form $z^n = r \text{ cis } \theta$, or $z^n = a + bi$ where a and b are real and n is a positive integer.

Solving Quadratic Equations by Completing the Square

Completing the Square

Not all quadratics factorise. Therefore we need a technique that will enable us to solve quadratics that do not factorise. The first technique we look at is called **completing the square.**

Consider the pattern

$$(x \pm a)^2 = x^2 \pm 2ax + a^2$$

This expression is called a perfect square and is the basis of completing the square.

The method requires us to rewrite the quadratic as a perfect square adjusting the constant term to match the given quadratic.

We then take the square root of both sides of the expression and make x the subject to obtain the solution(s).

Solving a quadratic by completing the square is the easiest method when you are expected to express the answer in surd form.

Completing the Square is the way to go if you want an exact answer.

The square of half the coefficient of the x term will complete a perfect square.

To find the constant when completing the square, square the 'a' of $(x \pm a)^2$ and add or subtract the required amount.

Example

Use completing the square to solve the quadratic equation

$$x^2 - 6x + 2 = 0$$

Put the constant on the right-hand side

$$x^2 - 6x = {}^{-}2$$

Complete the square with $a = {}^{-}3$ (half the $^{-}6$).

$$x^2 - 6x + ({}^{-}3)^2 = {}^{-}2 + ({}^{-}3)^2$$

$$x^2 - 6x + 9 = {}^{-}2 + 9$$

$$(x - 3)^2 = {}^{-}2 + 9$$

$$(x - 3)^2 = 7$$

Solve

$$x - 3 = \pm\sqrt{7}$$

$$x = 3 \pm \sqrt{7}$$

$$x = 3 + \sqrt{7} \text{ and } x = 3 - \sqrt{7}$$

Example

Use completing the square to solve the quadratic equation

$$2x^2 + 4x - 8 = 0$$

Put the constant on the right-hand side

$$2x^2 + 4x = 8$$

Common factor 2

$$2(x^2 + 2x) = 8$$

Divide both sides by 2

$$x^2 + 2x = 4$$

Complete the square with a = 1 (half of 2)

$$x^2 + 2x + (1)^2 = 4 + (1)^2$$

$$(x + 1)^2 = 4 + 1$$

$$(x + 1)^2 = 5$$

Solve

$$x + 1 = \pm\sqrt{5}$$

$$x = {}^{-}1 \pm \sqrt{5}$$

$$x = {}^{-}1 + \sqrt{5} \text{ and } x = {}^{-}1 - \sqrt{5}$$

Example

For $|z - 1| = |z - 3|$ describe the locus of z in the complex plane and obtain a Cartesian equation for the locus.

Solution

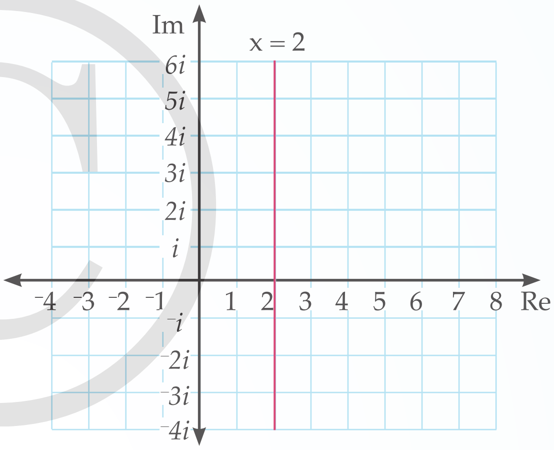
$$|z - 1| = |z - 3|$$

Substituting $z = x + yi$

we write

$$|(x + yi) - 1| = |(x + yi) - 3|$$
$$|(x - 1) + yi| = |(x - 3) + yi|$$
$$\sqrt{(x-1)^2 + y^2} = \sqrt{(x-3)^2 + y^2}$$
$$(x - 1)^2 + y^2 = (x - 3)^2 + y^2$$
$$x^2 - 2x + 1 + y^2 = x^2 - 6x + 9 + y^2$$
$$^-2x + 1 = {}^-6x + 9$$
$$4x = 8$$
$$x = 2$$

The locus of z is a vertical line through 2.



Example

For $|z| + |z - 2| = 6$ describe the locus of z geometrically and obtain a Cartesian equation for the locus.

Solution

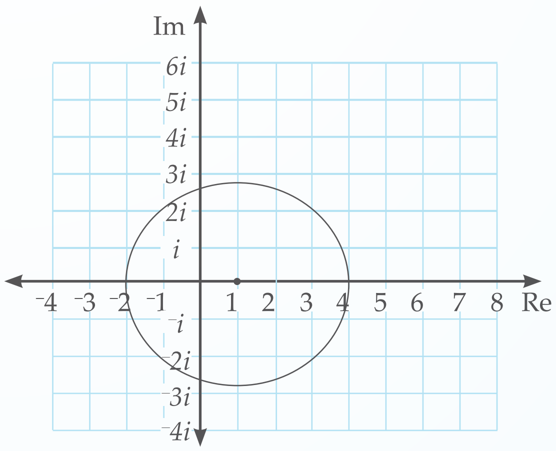
$$|z| + |z - 2| = 6$$

Substituting $z = x + yi$

we write

$$|(x + yi)| + |(x + yi) - 2| = 6$$
$$|(x + yi)| + |(x - 2) + yi| = 6$$
$$\sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 6$$
$$(x - 2)^2 + y^2 = (6 - \sqrt{x^2 + y^2})^2$$
$$x^2 - 4x + 4 + y^2 = 36 - 12\sqrt{x^2 + y^2} + x^2 + y^2$$
$$^-4x - 32 = {}^-12\sqrt{x^2 + y^2}$$
$$(^-4x - 32)^2 = 144(x^2 + y^2)$$
$$16x^2 + 256x + 1024 = 144x^2 + 144y^2$$
$$128x^2 - 256x + 144y^2 = 1024$$
$$128(x^2 - 2x) + 144y^2 = 1024$$
$$128(x - 1)^2 + 144y^2 = 1152$$
$$\frac{(x-1)^2}{9} + \frac{y^2}{8} = 1$$

The locus of z is an ellipse centre (1, 0) with major axis 6 and minor axis $2\sqrt{8}$.



Differentiation 3.6

This achievement standard involves applying differentiation methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply differentiation methods in solving problems.	• Apply differentiation methods, using relational thinking, in solving problems.	• Apply differentiation methods, using extended abstract thinking, in solving problems.

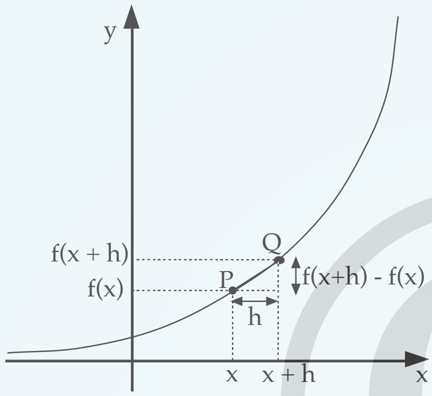
- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives
  - Identify discontinuities and limits of functions.
  - Choose and apply a variety of differentiation techniques to functions and relations using analytical methods.
- Apply differentiation methods in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using representations.
- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.
- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

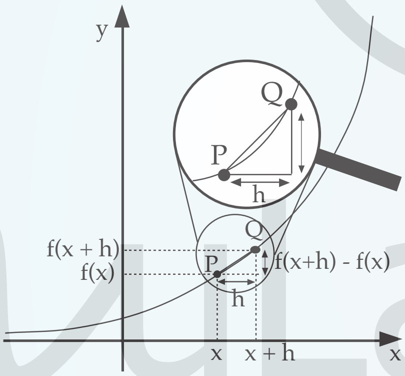
  and also using correct mathematical statements, or communicating mathematical insight.
- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.
- Methods include a selection from those related to:
  - derivatives of power, exponential, and logarithmic (base e only) functions
  - derivatives of trigonometric (including reciprocal) functions
  - optimisation
  - equations of normals
  - maxima and minima and points of inflection
  - related rates of change
  - derivatives of parametric functions
  - chain, product, and quotient rules
  - equations of normals
  - properties of graphs (limits, differentiability, continuity, concavity).

The Derived Function cont...

If we now reduce the value of h, the distance between the points P and Q will get smaller.



If we further reduce the value of h even more and find the limit as $h \to 0$, then the gradient of the chord PQ is essentially the gradient of a tangent at the point P.



As h approaches 0 the gradient of the chord approaches that of the tangent. Therefore we can find the gradient or slope of a tangent to a graph at any point, by using the formula

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x}$$

The notation $f'(x)$ is used to denote the gradient function (instead of the constant m) and $x + h - x$ simplifies to h.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This formula can be used to find the gradient at any point on a curve and also the function $f'(x)$ that describes the gradient of the curve.

The derived function is also known as the gradient function and the process of finding it is called **differentiation**.

When we use this formula to calculate the derived function we are **differentiating by first principles**.

Example

Differentiate by first principles the function

$f(x) = x^2 - 5$

We begin by finding an expression for

$f(x + h) = (x + h)^2 - 5$

$= x^2 + 2xh + h^2 - 5$

and then use the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We substitute for $f(x)$ and $f(x + h)$.

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 - 5) - (x^2 - 5)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

Dividing through by h gives

$$f'(x) = \lim_{h \to 0} 2x + h$$

$f'(x) = 2x$ as h tends to 0

Example

Differentiate by first principles the function

$f(x) = 3x^2 + 2x + 4$

We begin by finding an expression for

$f(x + h) = 3(x + h)^2 + 2(x + h) + 4$

$= 3x^2 + 6xh + 3h^3 + 2x + 2h + 4$

and then use the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 4) - (3x^2 + 2x + 4)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{6xh + 3h^2 + 2h}{h}$$

Dividing through by h gives

$$f'(x) = \lim_{h \to 0} 6x + 3h + 2$$

$f'(x) = 6x + 2$ as h tends to 0

The term 'by first principles' should alert you to use this approach in deriving the function.

Differentiation by first principles is a Merit concept of 'demonstrating understanding of concepts'.

Graphing Derived Functions

Graphing the Derived Function

To draw a sketch of the derived function f′(x) from a graph of the function f(x) we focus on key features of the function f(x) and interpret what these mean in terms of the derived function f′(x).

Features of the graph of f(x) to look out for and the corresponding features of f′(x) are:

1. the graph of f(x) is increasing ⟶ f′(x) > 0
2. the graph of f(x) is decreasing ⟶ f′(x) < 0
3. stationary points (maximum or minimum) of f(x) ⟶ x intercepts of f′(x)
4. points of inflection of f(x) ⟶ turning points of f′(x)
5. stationary points of inflection of f(x) ⟶ turning points on the x axis of f′(x)
6. vertical asymptotes of f(x) ⟶ vertical asymptotes of f′(x)
7. horizontal asymptotes of f(x) ⟶ horizontal asymptotes of f′(x)
8. spikes or discontinuities of f(x) ⟶ f′(x) is undefined.

A good technique is to begin by drawing a set of axes directly under a copy of y = f(x) so the scales on the x axis line up.

Next locate the stationary points (turning points) of f(x) which line up with the x intercepts of f′(x).

Then look for the points of inflection of f(x) which will line up with the turning points of f′(x).

Next identify where f(x) is increasing, which means that f′(x) is above the x axis.

Then identify where f(x) is decreasing which means that f′(x) is below the x axis.

Note any vertical or horizontal asymptotes of f′(x) and mark these in on f′(x).

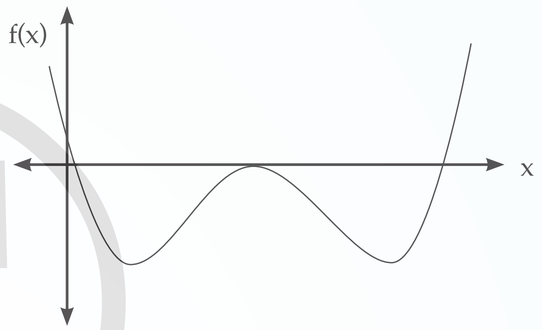
Note any ‘spikes’ or discontinuities (abrupt changes in the gradient) of f(x) as f′(x) will be undefined for these values.

Draw a smooth curve to fit this.

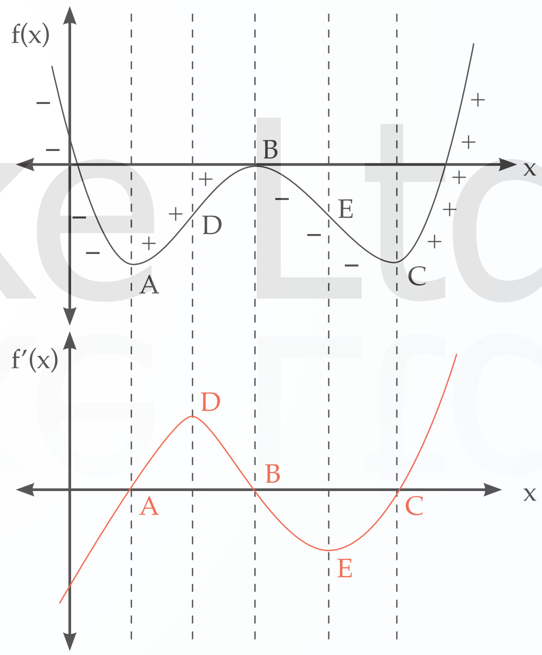
Study the example on the right and the following page to understand the process of graphing the derived function f′(x) when given a sketch of f(x).

Example

The graph of y = f(x) is drawn below. Sketch the graph of the derived function y = f′(x).



We begin by drawing a set of axes directly under a copy of y = f(x) so the scales on the x axis line up.



We then locate the stationary points (turning points) on y = f(x), labelled A, B and C. These become the x intercepts for y = f′(x).

There are two points of inflection on y = f(x), labelled D and E. These become turning points on y = f′(x).

Next we identify where f(x) is increasing (+), which means f′(x) is above the x axis.

Then we identify where f(x) is decreasing (–), which means f′(x) is below the x axis.

There are no vertical or horizontal asymptotes or discontinuities or ‘spikes’ on y = f(x).

We now draw a smooth curve to sketch y = f′(x).

Integration 3.7

This achievement standard involves applying integration methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply integration methods in solving problems.	• Apply integration methods, using relational thinking, in solving problems.	• Apply integration methods, using extended abstract thinking, in solving problems.

- This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives:
  - choose and apply a variety of integration and anti-differentiation techniques to functions and relations using both analytical and numerical methods
  - form differential equations and interpret the solutions

  in the Mathematics strand of the Mathematics and Statistics Learning Area.

- Apply integration methods in solving problems involves:
  - selecting and using methods
  - demonstrating knowledge of concepts and terms
  - communicating using appropriate representations.

- Relational thinking involves one or more of:
  - selecting and carrying out a logical sequence of steps
  - connecting different concepts or representations
  - demonstrating understanding of concepts
  - forming and using a model;

  and relating findings to a context, or communicating thinking using appropriate mathematical statements.

- Extended abstract thinking involves one or more of:
  - devising a strategy to investigate or solve a problem
  - identifying relevant concepts in context
  - developing a chain of logical reasoning, or proof
  - forming a generalisation;

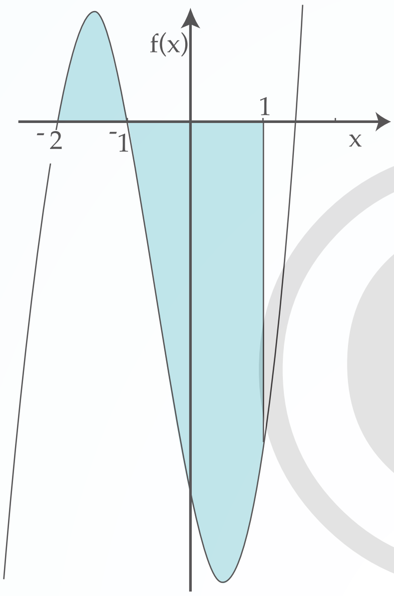
  and using correct mathematical statements, or communicating mathematical insight.

- Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

- Methods include a selection from those related to:
  - integrating power, polynomial, exponential (base e only), trigonometric, and rational functions
  - reverse chain rule, trigonometric formulae
  - rates of change problems
  - areas under or between graphs of functions, by integration
  - finding areas using numerical methods, e.g. the rectangle or trapezium rule
  - differential equations of the forms $y' = f(x)$ or $y'' = f(x)$ for the above functions or situations where the variables are separable (e.g. $y' = ky$) in applications such as growth and decay, inflation, Newton's Law of Cooling and similar situations.

Example

Find the area between the curve $f(x) = 2x^3 + 3x^2 - 5x - 6$ and the x axis from $x = {}^-2$ to $x = 1$.



Solution

Area = Area above + Area below

$$\text{Area} = \int_{-2}^{-1} 2x^3 + 3x^2 - 5x - 6\,dx + \left|\int_{-1}^{1} 2x^3 + 3x^2 - 5x - 6\,dx\right|$$

$$\text{Area} = \left[\frac{2}{4}x^4 + x^3 - \frac{5}{2}x^2 - 6x\right]_{-2}^{-1} + \left[\frac{2}{4}x^4 + x^3 - \frac{5}{2}x^2 - 6x\right]_{-1}^{1}$$

$$= \left(\frac{1}{2} - 1 - \frac{5}{2} + 6\right) - \left(\frac{16}{2} - 8 - \frac{20}{2} + 12\right) + \left(\frac{1}{2} + 1 - \frac{5}{2} - 6\right) - \left(\frac{1}{2} - 1 - \frac{5}{2} + 6\right)$$

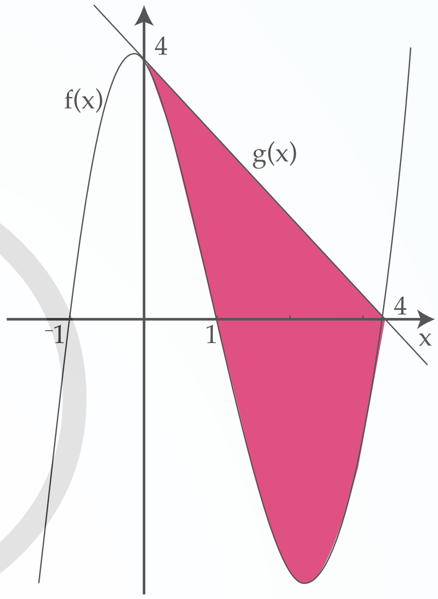
$= 3 - 2 + |{}^-7 - 3|$

$= 1 + 10$

$= 11$ units^2

Example

Find the area between the curve $f(x) = x^3 - 4x^2 - x + 4$ and the line $g(x) = 4 - x$ from $x = 0$ to $x = 4$.



Solution

Although the curve is above and below the x axes we are only interested in the enclosed area.

If we let $D(x) = g(x) - f(x)$

then as $g(x)$ is always above $f(x)$ from $x = 0$ to $x = 4$ then $D(x)$ is positive in this region.

Therefore we can integrate it to find the area enclosed.

$$\text{Area} = \int_0^4 D(x)dx$$

$$= \int_0^4 g(x) - f(x)dx$$

$$= \int_0^4 (4 - x) - (x^3 - 4x^2 - x + 4)dx$$

$$= \int_0^4 {}^-x^3 + 4x^2\,dx$$

$$= \left[\frac{{}^-x^4}{4} + \frac{4}{3}x^3\right]_0^4$$

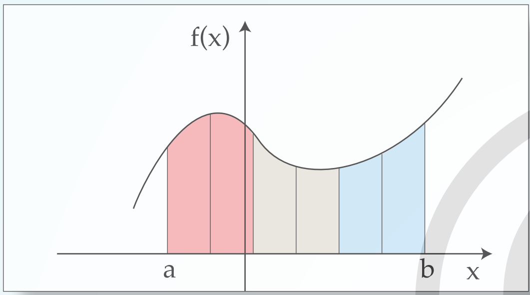
$$= 21\frac{1}{3} \text{ units}^2$$

It is important we set D(x) equal to the higher expression (g(x)) minus the lower expression f(x)) over the range otherwise the integral would be negative.

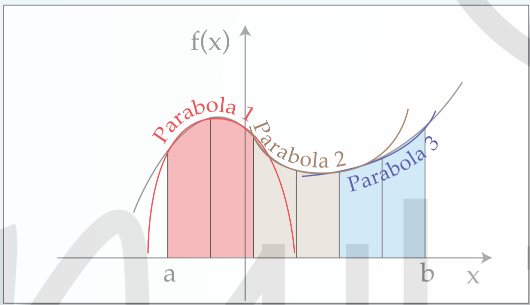
Simpson's Rule

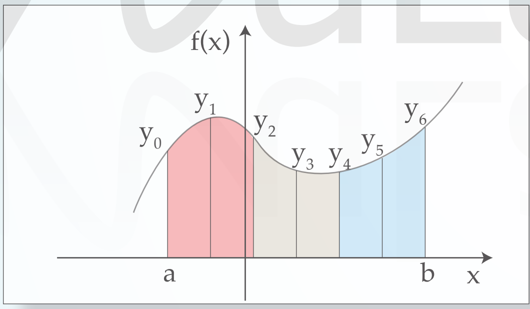
Simpson's Rule

Simpson's Rule models a definite integral problem by dividing the area up into sets of column pairs of constant width h.



Each of the pairs of columns is capped by a best fit parabola. The area of each pair of columns is calculated to find the total area.





Looking at just the first pair of columns (width h) the area is given by

$$\text{Area}_1 = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Now the area of all three pairs of columns becomes

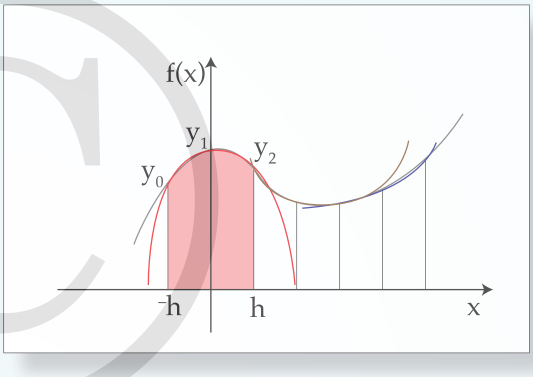
$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6)$$

$$\text{Area} = \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$\text{Area} = \frac{h}{3}(y_0 + y_6 + 4y_{\text{Odd}} + 2y_{\text{Even}})$$

Partial Derivation of Simpson's Rule

To derive the rule for the area of a pair of columns capped with a parabola we simplify the problem by having one pair either side of the y axes and then extend our answer. Let the y ordinates at the top of the first pair of columns be y_0, y_1 and y_2. The columns are each of width h.



We now derive a formula for the area under the parabola with equation $f(x) = Ax^2 + Bx + C$ passing through the points $(^-h, y_0)$, $(0, y_1)$ and (h, y_2).

Integrating f(x) to find the area of the two columns

$$\text{Area} = \int_{-h}^{h} Ax^2 + Bx + C\,dx$$

$$= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_{-h}^{h}$$

$$= \frac{2Ah^3}{3} + 2Ch$$

$$= \frac{h}{3}\left(2Ah^2 + 6C\right)$$

Since the points $(^-h, y_0)$, $(0, y_1)$ and (h, y_2) are on the parabola and $y_1 = C$ is the y intercept, they satisfy $f(x) = Ax^2 + Bx + C$.

So $\quad y_0 = A(^-h)^2 + B(^-h) + y_1$

$\quad y_0 = Ah^2 - Bh + C$

$\quad y_1 = C$

and $\quad y_2 = Ah^2 + Bh + C$

We note that $2Ah^2 + 6C = (Ah^2 - Bh + C) + 4C + (Ah^2 + Bh + C)$

$= y_0 + 4y_1 + y_2$

Therefore the area under the parabola is

$$= \frac{h}{3}(y_0 + 4y_1 + y_2)$$

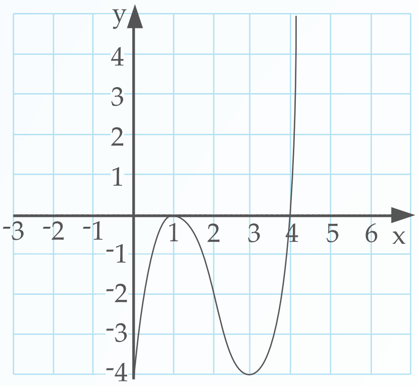
Adding more pairs of columns results in the standard Simpson's Rule. Moving the first pair of columns sideways does not change the generality but makes the maths more tedious. This derivation is not required for NCEA 3.

Page 136

288. a) Maximum (1, 0)
Minimum $(3, {}^{-}4)$

b) Inflection $(2, {}^{-}2)$

c)



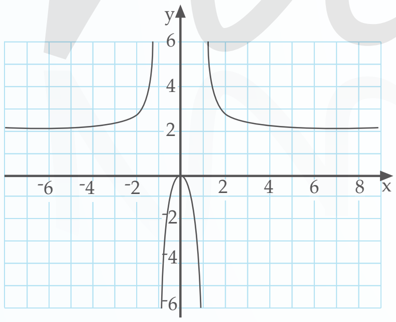
d) $x > 2$

289. a) $A = ({}^{-}0.645, 0)$
$B = ({}^{-}0.5, 0.25)$
$C = (0, 0)$
$D = (0.5, {}^{-}0.25)$
$E = (0.645, 0)$

b) At the points of inflection $x = 0, \pm 0.3536$
Gradient when $x = 0.3536$ is ${}^{-}0.938$ (3 sf)

290. a) Maximum point (0, 0)

b)



c) limit = 2

Page 139

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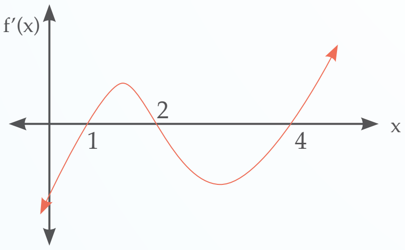

Page 139 cont...

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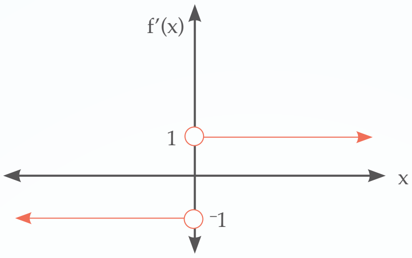

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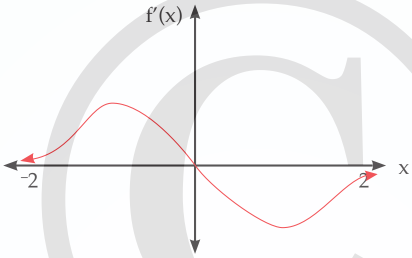

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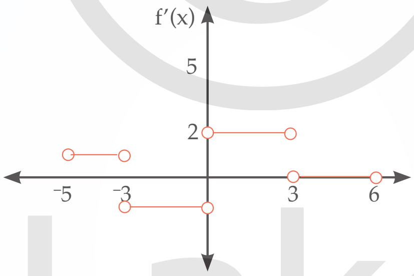

Page 140

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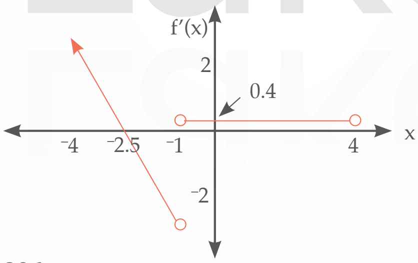
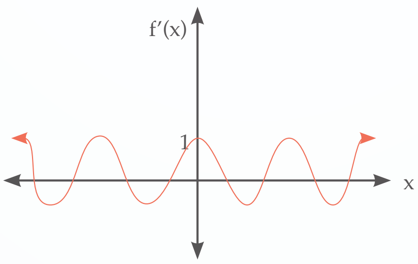


296.



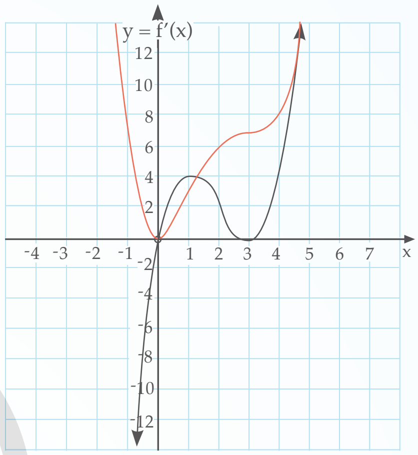
297. a) $x = 0$

b) $x = 1$ and 3

c) Stationary point of inflection because at $x = 3$, $y = f'(x)$ is both an intercept and a stationary point.

Page 140 cont...

297. d)



Page 142

298. $\dfrac{dy}{dx} = \dfrac{3}{10t}$

299. $\dfrac{dy}{dx} = \dfrac{t}{2}$

300. $\dfrac{dy}{dx} = \dfrac{{}^{-}2}{5}\tan t$

301. $\dfrac{dy}{dx} = \dfrac{\cos t}{e^t}$

302. $\dfrac{dy}{dx} = 4t\sqrt{t}$

303. $\dfrac{dy}{dx} = 2t^2(1 - t)$

304. $\dfrac{dy}{dx} = \dfrac{3t^2 - 1}{2t - 3}$

305. $\dfrac{dy}{dx} = \dfrac{2t^2 + 1}{t^2 - 1}$

306. $\dfrac{d^2y}{dx^2} = \dfrac{3}{4t}$

307. $\dfrac{d^2y}{dx^2} = \dfrac{{}^{-}5}{16\cos^3 t}$

Page 143

308. $\dfrac{d^2y}{dx^2} = {}^{-}3\sec^3 t$

309. $\dfrac{d^2y}{dx^2} = \dfrac{{}^{-}(t+1)}{t^2e^{2t}}$

310. $\dfrac{dy}{dx} = \dfrac{2}{2t+3}$

At $t = 0$ the tangent is $3y - 2x = 0$